23. RUMYANTSEV V.V., On the stability of stationary satellite motion, Vyschitel. Tsentr Akad. Nauk SSSR, 1967.

Translated by M.D.F.

PMM U.S.S.R.,Vol.48,No.5,pp. 630-634,1984
Printed in Great Britain.

0021-8928/84 $10.00+0.00

ON AN INTEGRAL EQUATION OF CONTACT PROBLEMS OF ELASTICITY THEORY IN THE PRESENCE OF ABRASIVE WEAR*

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An algorithm based on the method of matched asymptotic expansions and enabling one to avoid mathematical incorrectness is proposed for solving the integral equations of contact problems taking abrasive wear of the surfaces of contiguous bodies into account. An exact solution is written for the convolution type integral equation of the second kind with a logarithmic kernel in a semi-infinite interval in the class of continuous functions that vanish at infinity.

A mathematical inaccuracy is committed in solving the integral equations of contact problems of elasticity theory in the presence of abrasive wear (/1-4/, etc.). The quantity characterizing the contact pressure distribution law and have a singularity of the square-root type for $t=0$ at the ends of the contact domain /5/ was expanded in a Fourier series in the eigenfunctions of a certain self-adjoint completely continuous integral operator acting in a space of square-summable functions. However, as follows from the general theory of Fourier series in Hilbert spaces /6/, such a series will be known to be divergent in the norm of the space $L_2(-1, 1)$.

The approach proposed below enables one to avoid this mathematical incorrectness and in conjunction with the method in /7,8/ enable a solution of the contact problems mentioned to be constructed in the whole range of time variation. The closed solution of the convolution type integral equation of the second kind with logarithmic kernel in a semi-infinite interval can also be used to investigate contact problems for rough elastic bodies (or to study contact problems in the presence of thin elastic coatings) /9/ when the coefficient of the main term of the integral equation tends to zero.

1. The initial equations of the contact problem of elasticity theory for a linearly deformable base of general type in the presence of abrasive nwear can be written in the form /4/

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi,t)\,k\left(\frac{\xi-x}{\lambda}\right)d\xi=\gamma(t)-f(x)-\int_{0}^{t}\varphi(x,\tau)\,V(\tau)\,d\tau \tag{1.1}$$

$$(|x|\leqslant 1,\ 0\leqslant t\leqslant T<\infty)$$

$$P(t)=\int_{-1}^{1}\varphi(x,t)\,dx \tag{1.2}$$

The piecewise-smooth function $V(t)\geqslant 0\ (0\leqslant t\leqslant T)$ and the kernel $k(z)$ of the integral equation (1.1) is representable in the form

$$k(z)=\int_{0}^{\infty}L(u)\cos(uz)\,du,\quad z=\frac{\xi-x}{\lambda} \tag{1.3}$$

$$L(u)>0,\ \ (|u|<\infty),\ \ L(u)=A+O(u^2)\ \ (u\to 0,\ A=\text{const})$$

$$L(u)=u^{-1}+O(u^{-3})\ \ (u\to\infty)$$

The analysis presented below refers to the case of an even function $f(x)$. The general case is considered analogously.

On the basis of (1.3), the following lemma is proved /5/:

Lemma. For all values of $0\leqslant|z|<\infty$ the following representation holds for $k(z)$

*Prikl.Matem.Mekhan.,Vol.48,5 868-873,1984

$$k(z) = -\ln|z| - F(z), \quad F(z) \to B\ (z \to 0,\ B = \text{const})]$$

where $F(z)$, as an even function of the complex variable $w = z + i\zeta$, is regular in the strip $|z| < \infty$, $|\zeta| < \varkappa$ $(0 < \varkappa = \text{const})$.

We now construct the solution of (1.1). We introduce the small parameter $\varepsilon\ (\varepsilon \ll 1)$ and rewrite (1.1) in the form

$$\frac{1}{\pi}\int_{-1}^{1} \varphi(\xi, \varepsilon)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \gamma(\varepsilon) - f(x) - \int_0^{\varepsilon} \varphi(x, \tau) V(\tau)\, d\tau \quad (|x| \leqslant 1) \tag{1.4}$$

$$\frac{1}{\pi}\int_{-1}^{1} [\varphi(\xi, t) - \varphi(\xi, \varepsilon)]\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \gamma(t) - \gamma(\varepsilon) - \int_{\varepsilon}^{t} \varphi(x, \tau) V(\tau)\, d\tau \tag{1.5}$$

$$(|x| \leqslant 1,\ \varepsilon \leqslant t \leqslant T < \infty)$$

The solution of the integral equation (1.5) is found in /7,8/, hence, we shall not construct it here.

We investigate (1.4) by the method of matched asymptotic expansion /10/. We assume that

$$V(t) = V(\varepsilon) + O(\varepsilon), \quad \varphi(x, t) = \varphi(x, \varepsilon) + O(\varepsilon) \quad (t \in [0, \varepsilon],\ \varepsilon \ll 1) \tag{1.6}$$

Later, without loss of generality, we set $V(\varepsilon) = 1$. Then by substituting (1.6) into (1.4) and neglecting infinitesimals of order ε^2, we obtain

$$\varepsilon\varphi(x, \varepsilon) + \frac{1}{\pi}\int_{-1}^{1} \varphi(\xi, \varepsilon)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \gamma(\varepsilon) - f(x) \qquad (|x| \leqslant 1) \tag{1.7}$$

We shall limit ourselves to the construction of the principal (zeroth) term of the asymptotic form of the solution of (1.7). To do this, we consider its equivalent integral equation in place of (1.7)

$$\varepsilon[\varphi(x, \varepsilon) - \varphi(1, \varepsilon)] + \frac{1}{\pi}\int_{-1}^{1} \varphi(\xi, \varepsilon)\left[k\left(\frac{\xi - x}{\lambda}\right) - k\left(\frac{\xi - 1}{\lambda}\right)\right] d\xi = f(1) - f(x) \quad (|x| \leqslant 1) \tag{1.8}$$

Setting $\varepsilon = 0$ in (1.8), we write down the degenerate integral equation of the problem as $\varepsilon \to 0$

$$\int_{-1}^{1} \varphi_0(\xi)\left[k\left(\frac{\xi - x}{\lambda}\right) - k\left(\frac{\xi - 1}{\lambda}\right)\right] d\xi = \pi[f(1) - f(x)] \quad (|x| \leqslant 1) \tag{1.9}$$

or equivalently

$$\int_{-1}^{1} \varphi_0(\xi)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi[\gamma(0) - f(x)] \quad (|x| \leqslant 1) \tag{1.10}$$

As is known /5/, if $f(x) \in B_1^{\alpha}(-1, 1)$ ($B_1^{\alpha}(-1, 1)$ is the space of functions whose n-th derivatives satisfy the Hölder condition with index α, $0 < \alpha \leqslant 1$) for $x \in [-1, 1]$, the function $\varphi_0(x)$ has the form

$$\varphi_0(x) = \frac{\omega(x)}{\sqrt{1 - x^2}}, \quad \omega(x) \in C(-1, 1) \tag{1.11}$$

We understand the exterior domain to be the interval $|x| \leqslant 1 - m\varepsilon$ in which the degenerate solution of (1.11) can be taken as the solution of (1.8) with a sufficiently small error. We call small neighbourhoods of the points $x = \pm 1$ with the dimensions $m\varepsilon\ (m \geqslant 1)$ interior domains; the influence of the wear on the contact stress distribution under the stamp in these domains is commensurate with the influence of the deformability of the elastic base. Solutions of boundary-layer type that can be matched with the degenerate solution $\varphi_0(x)$ on the domain boundaries $x = 1 - m\varepsilon$ and $x = -1 + m\varepsilon$ should be constructed in the interior domains.

Let us find the difference between (1.8) and (1.9)

$$\varepsilon\varphi(x, \varepsilon) + \frac{1}{\pi}\int_{-1}^{1} [\varphi(\xi, \varepsilon) - \varphi_0(\xi)]\left[k\left(\frac{\xi - x}{\lambda}\right) - k\left(\frac{\xi - 1}{\lambda}\right)\right] d\xi = \varepsilon\varphi(1, \varepsilon) \quad (|x| \leqslant 1) \tag{1.12}$$

Taking into account that we have $\varphi_0(1 - m\varepsilon) = \omega(1)(2m\varepsilon)^{-1/2}$ for $m\varepsilon \ll 1$ and the function $\varphi(x, \varepsilon)$ is matched with $\varphi_0(x)$ on the boundary $x = 1 - m\varepsilon$ of the exterior and interior domains, we will seek the solution of boundary-layer type in the neighbourhood of the point $x = 1$ in the form of

$$\varphi(x,\varepsilon)=\psi(s)\,\varepsilon^{-1/2}+o(\varepsilon^{-1/2}),\quad s=(1-x)\,\varepsilon^{-1} \tag{1.13}$$

Then the matching conditions take the form

$$\varphi(1-m\varepsilon,\varepsilon)=\varepsilon^{-1/2}q(m)\,\psi(0)\sim\varphi_0(1-m\varepsilon)=\omega(1)\,(2m\varepsilon)^{-1/2}$$

It hence follows that

$$q(0)\sim 1,\quad q(m)\sim m^{-1/2}\ (m\geqslant 1),\quad \psi(0)=\omega(1)/\sqrt{2} \tag{1.14}$$

Substituting (1.13) into (1.12), going over to the new variables $s,\tau=(1-\xi)\,\varepsilon^{-1}$ and letting ε tend to zero for fixed s and $\varepsilon\ll\lambda$, we obtain the following integral equation to determine the boundary-layer type function $\psi(s)$ (by virtue of the evenness of the function under consideration in x and in the neighbourhood of the point $x=-1$)

$$q(s)-\frac{1}{\pi}\int_0^\infty q(\tau)\ln\left|\frac{\tau-s}{\tau}\right|d\tau=1,\quad q(s)=\frac{\psi(s)}{\psi(0)}\quad(0\leqslant s<\infty) \tag{1.15}$$

Here the representation for $k(z)$ mentioned in the lemma and the value of the integral /11/

$$\int_0^\infty \ln\left|1-\frac{s}{\tau}\right|\tau^{-p}\,d\tau=\pi s^{1-p}\,\frac{\operatorname{ctg}\pi p}{p-1}\quad(0<\operatorname{Re}p<1)$$

for $p={}^1/_2$ are used.

After having solved (1.10) and (1.15), by virtue of (1.14) the principal term of a uniformly suitable asymptotic solution of the integral equation (1.7) or (1.8) can be represented for small values of the parameter ε in the form

$$\varphi_u(x,\varepsilon)=\frac{1}{\sqrt{1-x^2}}\left[\omega(x)-\frac{\omega(1)}{2}\left(\sqrt{1+x}+\sqrt{1-x}\right)\right]+\frac{\omega(1)}{\sqrt{2\varepsilon}}\left[q\left(\frac{1+x}{\varepsilon}\right)+q\left(\frac{1-x}{\varepsilon}\right)\right] \tag{1.16}$$

The constant $\omega(1)$ can here be associated with $\gamma(0)$ or with $P(0)$ by using (1.10),(1.11) or (1.2) for $t=0$.

2. We will examine the questions of constructing the solution of the integral equation (1.15). We differentiate both sides with respect to s. We obtain

$$q'(s)+\frac{1}{\pi}\int_0^\infty q(\tau)\frac{d\tau}{\tau-s}=0\quad(0\leqslant s<\infty) \tag{2.1}$$

We seek the solution of the homogeneous equation (2.1) in the form of a Mellin integral /11/ (the contour L is the line $\operatorname{Re}p=\mu$)

$$q(s)=\frac{1}{2\pi i}\int_L Q(p)\,s^{-p}\,dp \tag{2.2}$$

Let us substitute (2.2) into (2.1) and take the value of the integral

$$\int_0^\infty\frac{\tau^{-p}}{\tau-s}\,d\tau=\pi s^{-p}\operatorname{ctg}\pi p\quad(0<\operatorname{Re}p<1)$$

into account.

We then rewrite (2.1) in the form

$$\frac{1}{2\pi i}\int_L pQ(p)\,s^{-p-1}\,dp-\frac{1}{2\pi i}\int_L Q(p)\operatorname{ctg}\pi p\,s^{-p}\,dp=0 \tag{2.3}$$

We use the notation $pQ(p)=u(p)$, replace the argument p in the first integral of (2.3) by $p-1$, shift the contour L by one along the real axis and denote it by L_1. We will have

$$\frac{1}{2\pi i}\int_{L_1}u(p-1)\,s^{-p}\,dp=\frac{1}{2\pi i}\int_L u(p)\,\frac{\operatorname{ctg}\pi p}{p}\,s^{-p}dp \tag{2.4}$$

We assume that the function $u(p)$ is regular in the strip $\mu-1\leqslant\operatorname{Re}p\leqslant\mu$ and tends to zero as $|\operatorname{Im}p|\to\infty$. Then by the Cauchy theorem, without changing the integrand in the first integral of (2.4), it is possible to write L in place of L_1 and to satisfy relationship (2.4) by solving the first-order difference equation

$$u(p-1)-p^{-1}\operatorname{ctg}\pi p\,u(p)=0\quad(p\in L) \tag{2.5}$$

Let us select the number μ in such a way that the coefficient of (2.5) (i.e., the function $p^{-1}\operatorname{ctg}\pi p$) would have no zeros $(0<\mu<{}^1/_2)$ in the strip $0<\operatorname{Re}p<\mu$. In this case the canonical solution of the homogeneous equation (2.5) can be obtained by the Barnes method /12/.

The uniqueness of the solutions of difference equations of the type (2.5) is established* in exactly the same way as the uniqueness of the Riemann boundary value problem /13/. Taking account of the condition imposed above on the function $u(p)$, we write it in the form /14/

$$u(p) = C\Gamma^2(1+p)R(p) \quad (C = \mathrm{const}) \tag{2.6}$$
$$R(p) = \prod_{n=1}^{\infty}\left(1-\frac{1}{2n}\right)^{2p+1}\frac{\Gamma(-n-p+{}^1/_2)\Gamma(-n+p+1)}{\Gamma(-n-p)\Gamma(-n+p+{}^3/_2)}, \quad u(0)=C$$

from which we find the function $Q(p)$ by virtue of the relation presented earlier between $Q(p)$ and $u(p)$.

Let us set up the asymptotic form of the desired function $q(s)$ in (2.2) as $s\to 0$ and $s\to\infty$. We note that for $s<1$ the integral (2.2) is expanded in an absolutely convergent series in residues at the poles of the function $Q(p)$, continued analytically into the left half-plane $\operatorname{Re} p \leqslant \mu$. The value of the function $q(s)$ as $s\to 0$ is determined by the first term of this series and has the form $q(0)=C$.

For $s>1$ the integral (2.2) is not expanded in a residue series. In this case the asymptotic form of the function $q(s)$ is established for large values of s by using an algorithm proposed in /15/. Namely, by limiting ourselves to the first term in the asymptotic form, we obtain from (2.2) and (2.6)

$$q(s) = C\delta s^{-1/2} + O(s^{-1}) \tag{2.7}$$
$$\delta = -C^{-1}\lim \operatorname{tg}\pi p\,(p+{}^1/_2)\,u(p) = 1 \quad (p\to -{}^1/_2)$$

The solution of (2.2),(2.6) evidently satisfies the initial equation (1.15) apart from a constant. However, by utilizing the arbitrariness of the selection of C, it may be that this solution will simultaneously also be the solution of the integral equation (1.15).

Theorem. If $q(s)\sim s^{-1/2}\ (s\to\infty)$ and $|q(s)|<M\ (M=\mathrm{const})$ in (1.15) for all values $s\in[0,\infty)$, then $q(0)=1$.

For the proof we estimate the integral

$$J = -\int_0^\infty q(\tau)\ln\left|1-\frac{s}{\tau}\right|d\tau = -\int_0^{M_1} q(\tau)\ln\left|1-\frac{s}{\tau}\right|d\tau - \int_{M_1}^\infty q(\tau)\ln\left|1-\frac{s}{\tau}\right|d\tau = J_1+J_2 \tag{2.8}$$

for small values of the variable s. We assume that the constant M_1 in (2.8) is such that $q(\tau)\sim\tau^{-1/2}$ in J_2. Then if $s\to 0$, we have

$$J_2 \sim -s\int_{M_1}^\infty \frac{d\tau}{\tau^{3/2}} = -\frac{2s}{\sqrt{M_1}}$$

Furthermore, representing J_1 in the form

$$J_1 = -\int_0^s q(\tau)\ln\left(\frac{s}{\tau}-1\right)d\tau - \int_s^{M_1} q(\tau)\ln\left(1-\frac{s}{\tau}\right)d\tau = J_3+J_4$$

and using the estimate

$$|J_3| \leqslant 2Ms\ln 2, \quad |J_4| \leqslant Ms\,[1+\ln(M_1/s)]$$

we find that $J\to 0$ as $s\to 0$, and therefore, $q(0)=1$.

It follows from the theorem proved that the constant is $C=1$, and therefore, the function (2.2),(2.6) satisfies integral equation (1.15). We note that for such a selection of the arbitrary constant, the matching conditions (1.14) are satisfied automatically. Therefore, (2.2),(2.6) allow solution of the integral equation (1.15). This solution will be general in the class of functions possessing the properties mentioned in the theorem if the homogeneous integral equation corresponding to (1.15) has only a trivial solution. This last fact can be verified by taking account of the mutual relationships between the integral equation (1.15) and the problems examined in /14/.

3. We still note that the two fundamental variations of problem (1.1),(1.2) are of interest in practice: 1) given the function $\gamma(t)$, find $\varphi(x,t)$, $P(t)$, and 2) given $P(t)$, find the functions $\varphi(x,t)$ and $\gamma(t)$.

We will examine the first case. We assume that the settling of the points of the base $\gamma(t)$ is such that in the neighbourhood of the point $t=0$ it can be expanded in a Taylor

*Bantsuri, R.D. Contact problems of plane elasticity theory and associated boundary value problems of function theory. Doctorate Dissertation, Institute of Mechanics Problems, Academy of Sciences of the USSR, Moscow, 1981.

series. Limiting ourselves to the first two terms of this series, we will have

$$\gamma(t) = \gamma(0) + \gamma'(0)\, t \quad (t \in [0, \varepsilon]) \tag{3.1}$$

Now substituting the function $\varphi_u(x, t)$ from (1.16) and $\gamma(t)$ from (3.1) into the integral equation (1.7), we rewrite the latter in the form

$$\varepsilon\varphi_u(x, \varepsilon) + \frac{1}{\pi}\int_{-1}^{1} \varphi_u(\xi, \varepsilon)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \gamma(0) + \gamma'(0)\varepsilon - f(x) \quad (|x| \leqslant 1) \tag{3.2}$$

As was noted above, the constant $\omega(1)$ in (1.16) can be related to $\gamma(0)$ by means of (1.10), (1.11). Therefore, (see (1.2) for $t = 0$), the value of the function $P(t)$ depends only on $\gamma(0)$ for $t = 0$. We substitute (1.16) into relationship (3.2), integrate it with respect to x between the limits -1 and $+1$ and use (1.2), (1.11), (1.16). We obtain

$$\varepsilon P(\varepsilon) + \frac{1}{\pi}\int_{-1}^{1} \varphi_u^{1}(\xi, \varepsilon)\, D(\xi)\, d\xi = \gamma'(0)\varepsilon, \quad D(\xi) = \int_{-1}^{1} k\left(\frac{\xi - x}{\lambda}\right) dx \tag{3.3}$$

$$\varphi_u^{1}(x, \varepsilon) = -\frac{\omega(1)}{2}\left(\frac{1}{\sqrt{1-x}} + \frac{1}{\sqrt{1+x}}\right) + \frac{\omega(1)}{\sqrt{2\varepsilon}}\left[q\left(\frac{1+x}{\varepsilon}\right) + q\left(\frac{1-x}{\varepsilon}\right)\right]$$

We therefore establish a relationship between $P(\varepsilon)$ and $\gamma'(0)$.

Now, if its is assumed that the force $P(t)$ acting on the stamp is given, then by using (1.2) at $t = 0$ and (1.11), we find the value of the constant $\omega(1)$ by relating it to the value $P(0)$. Then substituting (1.11) into (1.10), we determine the constant $\gamma(0)$ by relating it, as above, to $P(0)$. Now using relationship (3.3), we find $\gamma'(0)$ as before by relating it to the value of $P(t)$ for $t = \varepsilon$. Finally, taking (3.1) in the computation, we obtain an expression to determine rigid displacement of the stamp $\gamma(t)$. Moreover, if (1.5) and the algorithm in /7,8/ are furthermore used, the solution of the problem formulated can be obtained in the whole range of time variation, i.e., for $0 \leqslant t \leqslant T < \infty$.

The method in this paper can also be used to investigate contact problems for rough elastic bodies (or contact problems in the presence of thin coatings /9/, when the coefficient of the principal term in the corresponding integral equation tends to zero).

The author is grateful to V.M. Aleksandrov, B.M. Nuller and M.B. Ryvkin for their interest.

REFERENCES

1. GALIN L.A., Contact problems of elasticity theory in the presence of wear, PMM, Vol. 40, No.6, 1976.
2. ALEKSANDROV V.M., GALIN L.A. and PIRIEV N.P., A plane contact problem for an elastic layer of considerable thickness in the presence of wear, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.4, 1978.
3. GALIN L.A. and GORYACHEVA I.G., Axisymmetric contact problem of elasticity theory in the presence of wear, PMM, Vol.41, No.5, 1977.
4. ALEKSANDROV V.M. and KOVALENKO E.V., Plane contact problems of elasticity theory for non-classical domains in the presence of wear, Prikl. Mekhan. Tekh. Fiz., No.3, 1980.
5. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical mixed problems of elasticity theory, Nauka, Moscow, 1974.
6. SUETIN P.K., Classical orthogonal polynomials, Nauka, Moscow, 1976.
7. ALEKSANDROV V.M. and KOVALENKO E.V., On a class of integral equations of mixed problems of the mechanics of continuous media, Dokl. Akad. Nauk SSSR, Vol.252, No.2, 1980.
8. KOVALENKO E.V., On the approximate solution of one type of integral equations of elasticity theory and mathematical physics, Izv. Akad. Nauk ArmSSR, Mekhanika, Vol.34, No.5, 1981.
9. KOVALENKO E.V., On an effective method of solving contact problems for a linearly deformable base with thin reinforcing coating, Izv. Akad. Nauk ArmSSR, Mekhanika, Vol.32, No.2, 1979.
10. NAYFEH A.H., Perturbation methods /Russian translation/, Mir, Moscow, 1976.
11. MARICHEV O.I., Method of evaluating integrals of special functions (theory and tables of formulas), Nauka, i Tekhnika, Minsk, 1978.
12. MIROLYUBOV A.A. and SOLDATOV M.A., Linear homogeneous difference equations Nauka, Moscow, 1981.
13. GAKHOV F.D., Boundary value problems, Nauka, Moscow, 1977.
14. NULLER B.M. and SPESIVTSEVA L.F., On a class of problems for elastic domains weakened by an inhomogeneous inclusion, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No. 2, 1979.
15 POPOV G. YA. and TIKHONENKO L.YA., Exact solution of plane problems on the contact of semi-infinite beams with an elastic wedge, PMM, Vol.39, No.6, 1975.

Translated by M.D.F.